3 Frequency Domain Representation of Continuous Signals and Systems

3.1 Fourier Series Representation of Periodic Signals 2
3.1.1 Exponential Fourier Series 2
3.1.2 Discrete Fourier Spectrum / Line Spectrum 3
3.1.3 Parseval's Theorem for Periodic Signals 7
3.2 Fourier Transform 9
3.2.1 Definition and Examples 9
3.2.2 Properties of the Fourier Transform 12
3.2.3 Symmetry of the Fourier Transform 20
3.2.4 Convolution 22
3.2.5 Parseval's Theorem for Energy Signals 24
3.2.6 Fourier Transform of Periodic Signals 25
3.3 Frequency Domain Description of LTI Systems 27
3.3.1 Frequency Response 27
3.3.2 Bandwidth of Frequency Selective Systems 30
3.3.3 Distortionless Transmission 31

3.1 Fourier Series Representation of Periodic Signals

3.1.1 Exponential Fourier Series

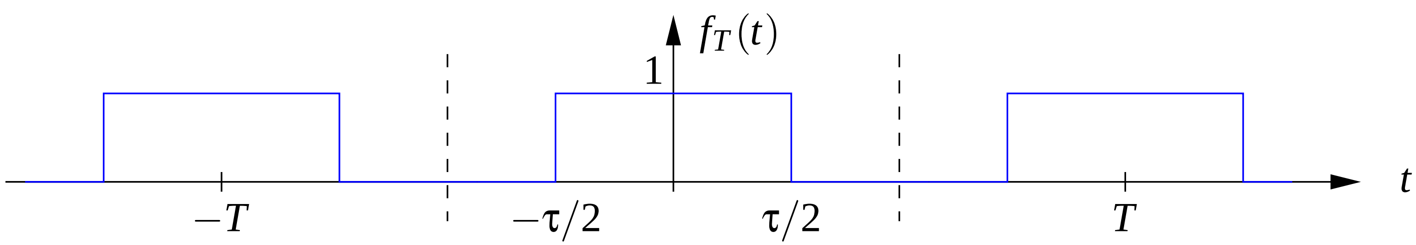
A large class of periodic signals $f_T(t)$ with period T and fundamental frequency $\omega_0 = 2\pi/T$ can be represented as a sum of harmonic complex exponential functions:

$$f_T(t) = \sum_{k=-\infty}^{\infty} F_k \, \exp(jk\omega_0 \, t)$$

with complex Fourier coefficients:

$$F_k = \frac{1}{T} \int_{t_0}^{t_0+T} f_T(t) \, \exp(-jk\omega_0 \, t) \, dt$$

Example: Rectangular Pulse Train



Calculation of Fourier coefficients F_k:

$$F_k = \frac{1}{T} \int_{-\tau/2}^{+\tau/2} \exp(-jk\omega_0\, t)\, dt = \frac{1}{T} \left[\frac{\exp(-jk\omega_0\, t)}{-jk\omega_0} \right]_{-\tau/2}^{\tau/2}$$

$$= \frac{1}{k\omega_0\, T} \cdot \frac{\exp(-jk\omega_0\, \tau/2) - \exp(jk\omega_0\, \tau/2)}{-j} = \frac{2\sin(k\omega_0\, \tau/2)}{k\omega_0\, T}$$

$$= \frac{\tau}{T} \cdot \frac{\sin(k\omega_0\, \tau/2)}{k\omega_0\, \tau/2} = \frac{\tau}{T} \cdot \mathrm{Sa}(k\omega_0\, \tau/2) = \frac{\tau}{T} \cdot \mathrm{Sa}(k\pi\tau/T)$$

$\mathrm{Sa}(x) = \sin(x)/x$: sine-over-argument function

3.1.2 Discrete Fourier Spectrum / Line Spectrum

Definition:

Graph of the (complex) Fourier coefficients F_k as a function of the angular frequency ω.

$$f_T(t) = \sum_{k=-\infty}^{\infty} F_k \, \exp(jk\omega_0\, t)$$

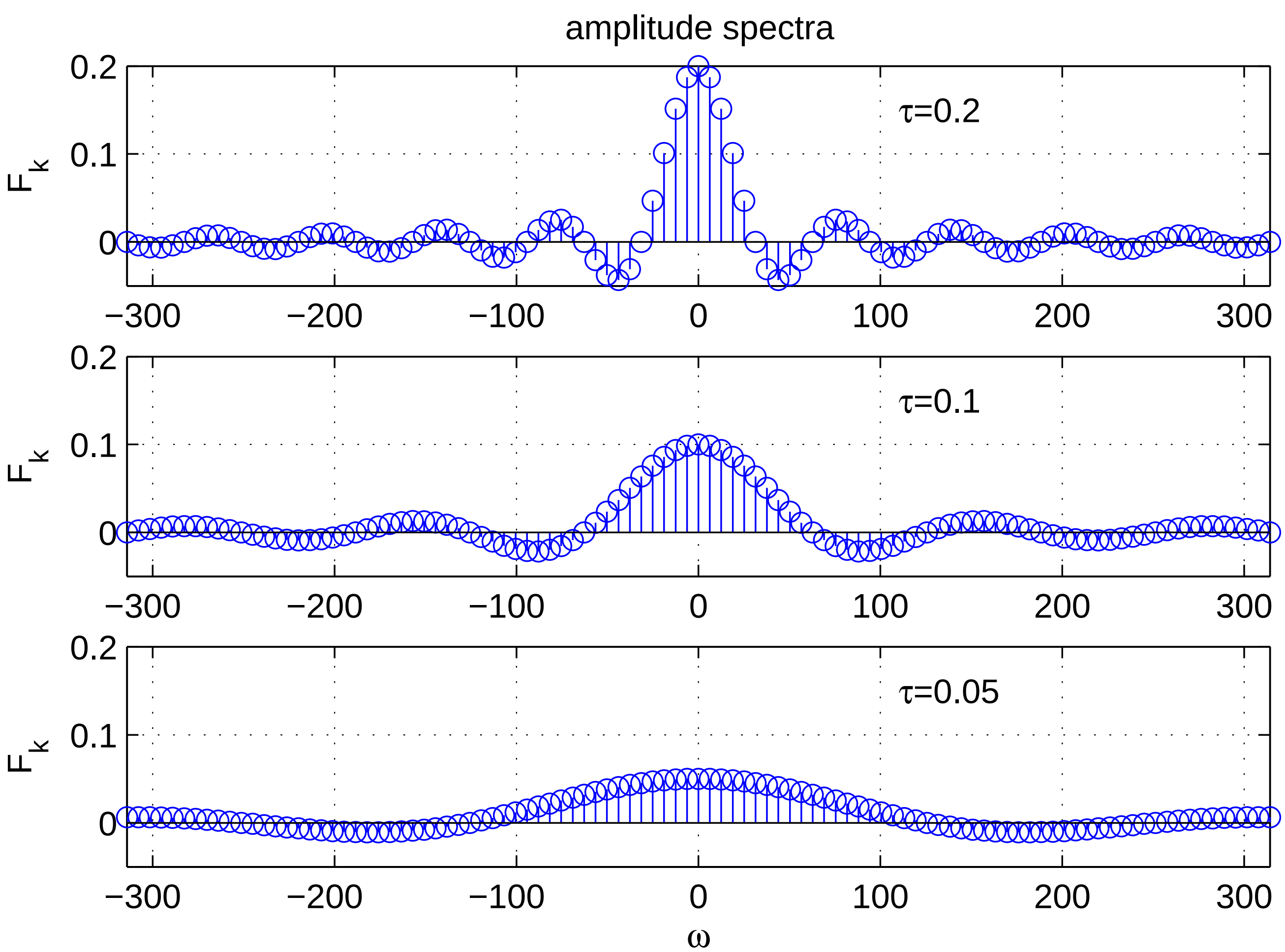
For a periodic signal, the Fourier spectrum exists only at discrete values of ω:

$\omega = 0,\ \pm\omega_0,\ \pm 2\omega_0,\ \pm 3\omega_0,\ \ldots$

Example: Rectangular Pulse Train

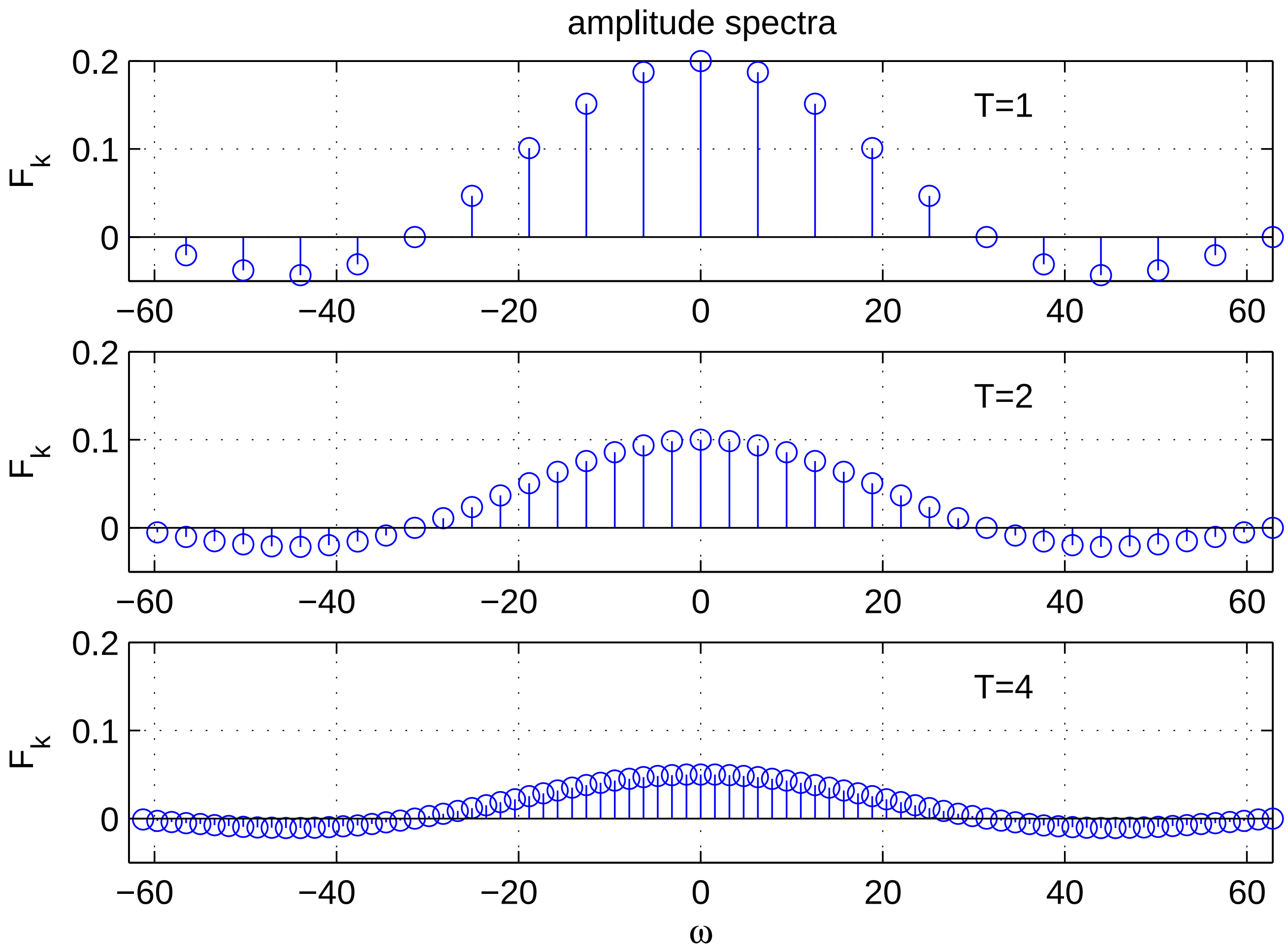
$T = 1$, τ varies:

$$f_T(t) = \begin{cases} 1 & |t| < \tau/2 \\ 0 & \tau/2 < |t| < 1/2 \end{cases}, \qquad F_k = \tau \cdot \frac{\sin(k\pi\tau)}{k\pi\tau} = \tau \cdot \mathsf{Sa}(k\pi\tau)$$



$\tau = 0.2$, T varies:

$$f_T(t) = \begin{cases} 1 & |t| < 0.1 \\ 0 & 0.1 < |t| < T/2 \end{cases}, \qquad F_k = \frac{0.2}{T} \cdot \mathrm{Sa}(0.2k\pi/T)$$



Magnitude and Phase Spectra:

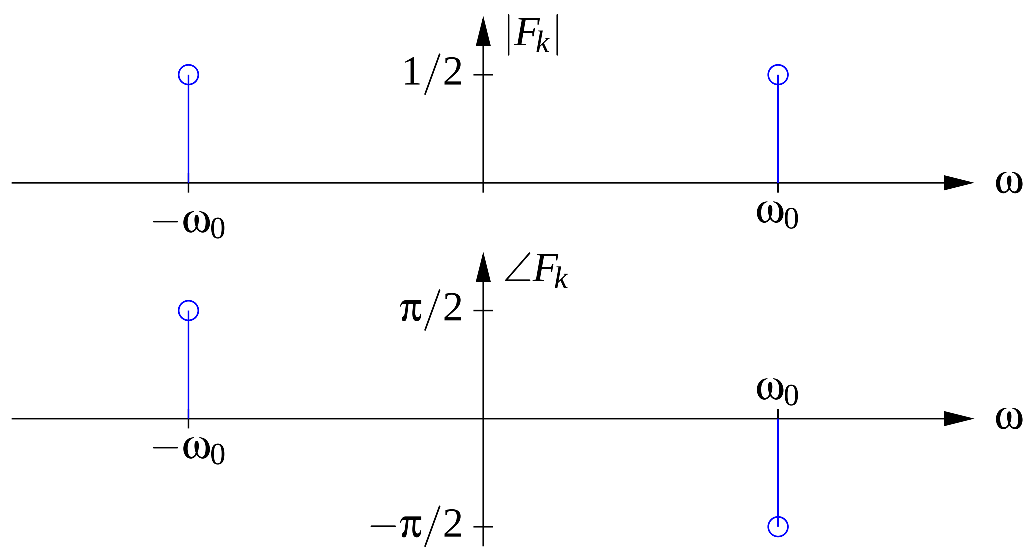
For complex Fourier coefficients F_k the magnitude and phase is generally plotted separately resulting in the magnitude and phase spectrum.

Example: Sinusoids

$$f_T(t) = \sin(\omega_0 t) = \underbrace{\frac{1}{2j}}_{F_1} \exp(j\omega_0 t) + \underbrace{\frac{-1}{2j}}_{F_{-1}} \exp(-j\omega_0 t)$$

$$F_1 = \frac{1}{2j} = \frac{1}{2}(-j) = \frac{1}{2}\exp(-j\pi/2) = \frac{1}{2}\angle - \pi/2$$

$$F_{-1} = -\frac{1}{2j} = \frac{1}{2}j = \frac{1}{2}\exp(j\pi/2) = \frac{1}{2}\angle\pi/2$$



3.1.3 Parseval's Theorem for Periodic Signals

A periodic signal $f_T(t)$ is a power signal with average power:

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} |f_T(t)|^2 \, dt = \frac{1}{T}\int_{t_0}^{t_0+T} f_T(t) f_T^*(t) \, dt$$

Exponential Fourier Series of complex conjugate function:

$$f_T^*(t) = \left(\sum_{k=-\infty}^{\infty} F_k \cdot \exp(jk\omega_0 \, t)\right)^* = \sum_{k=-\infty}^{\infty} F_k^* \cdot \exp(-jk\omega_0 \, t)$$

Average Power:

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} f_T(t) f_T^*(t) \, dt = \frac{1}{T}\int_{t_0}^{t_0+T} f_T(t) \cdot \sum_{k=-\infty}^{\infty} F_k^* \exp(-jk\omega_0 \, t) \, dt$$

$$= \sum_{k=-\infty}^{\infty} F_k^* \cdot \underbrace{\left[\frac{1}{T}\int_{t_0}^{t_0+T} f_T(t) \exp(-jk\omega_0 \, t) \, dt\right]}_{F_k} = \sum_{k=-\infty}^{\infty} |F_k|^2$$

Example: Sinusoids

$$f_T(t) = \sin(\omega_0 t) = \underbrace{\frac{1}{2j}}_{F_1} \exp(j\omega_0 t) + \underbrace{\frac{-1}{2j}}_{F_{-1}} \exp(-j\omega_0 t)$$

Average Power:

$$P = \frac{1}{T}\int_{-T/2}^{T/2} \sin^2(\omega_0 t)\, dt = |F_{-1}|^2 + |F_1|^2 = \frac{1}{2}$$

3.2 Fourier Transform

3.2.1 Definition and Examples

Frequency representation of an aperiodic signal.

Fourier Transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-j\omega t) dt$$

Inverse Fourier Transform:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(j\omega t)\, d\omega$$

- The Fourier Transform is taken over all times, i.e. all time resulution is lost.
- $F(\omega)$ is called Fourier Transform/spectral-density function/(Fourier) spectrum of $f(t)$. It is generally a complex valued function.
- Each point of $F(\omega)$ indicates the relative weighting of each frequency.
- Short-hand notation:

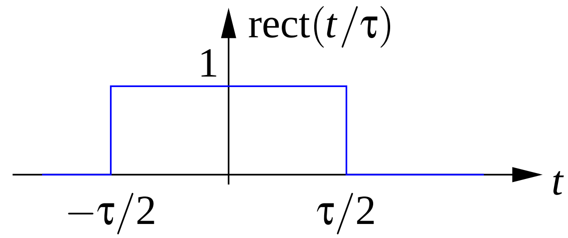
$$F(\omega) = \mathcal{F}\{f(t)\}, \qquad F(\omega) \bullet\!\!-\!\!\circ\, f(t), \qquad f(t) \circ\!\!-\!\!\bullet\, F(\omega)$$

- Sufficient condition for the existence of the Fourier Transform:

$$\int_{-\infty}^{\infty} |f(t)| dt < \infty$$

Example: Gate Function

$$f(t) = \mathsf{rect}(t/\tau)$$



Fourier Transform:

$$F(\omega) = \int_{-\infty}^{\infty} \mathsf{rect}(t/\tau) \cdot \exp(-j\omega t) dt = \int_{-\tau/2}^{\tau/2} \exp(-j\omega t) dt$$

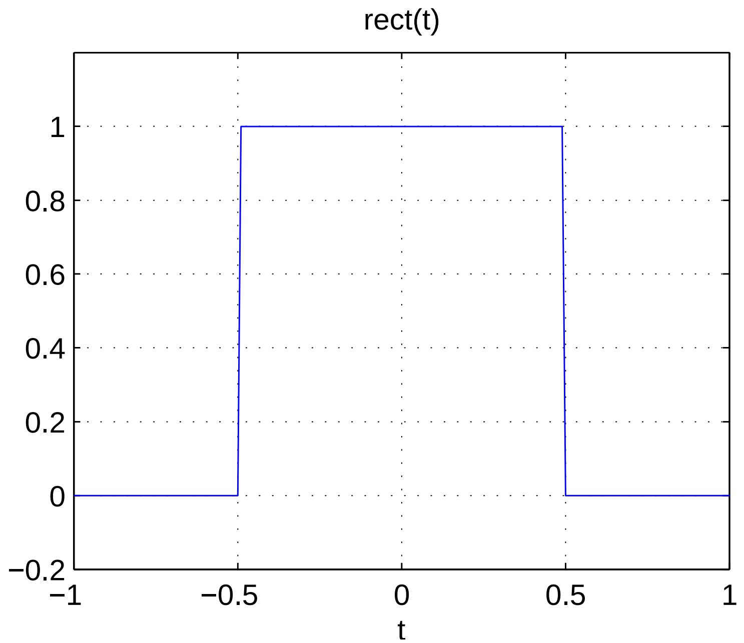
$$= \left[\frac{1}{-j\omega} \exp(-j\omega t) \right]_{-\tau/2}^{\tau/2} = -\frac{1}{j\omega} \left(\exp(-j\omega\tau/2) - \exp(j\omega\tau/2) \right)$$

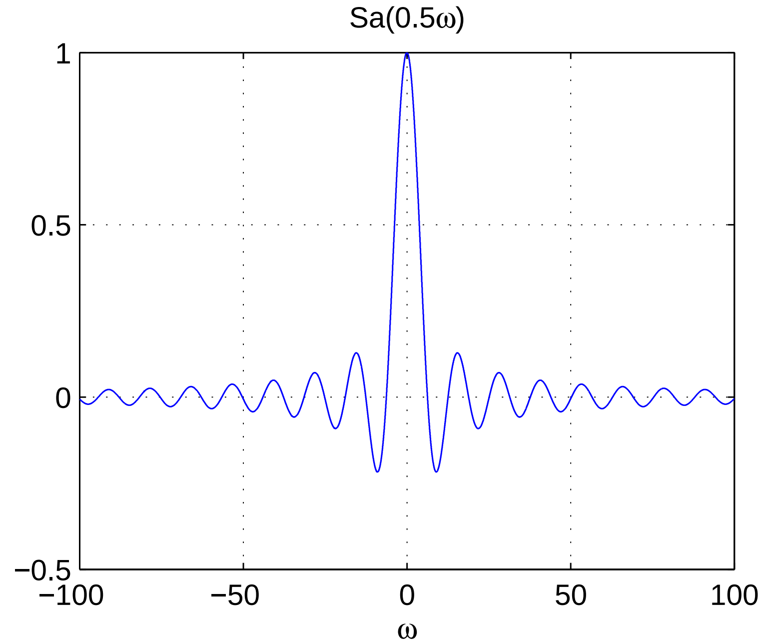
Euler's Identity:

$$F(\omega) = -\frac{\cos(-\omega\tau/2) + j\sin(-\omega\tau/2) - \cos(\omega\tau/2) - j\sin(\omega\tau/2)}{j\omega}$$

$$= \frac{2\sin(\omega\tau/2)}{\omega} = \tau \frac{\sin(\omega\tau/2)}{\omega\tau/2} = \tau\, \mathsf{Sa}(\omega\tau/2)$$

$$\mathsf{rect}(t/\tau) \circ\!\!-\!\!\bullet\ \tau\, \mathsf{Sa}(\omega\tau/2)$$





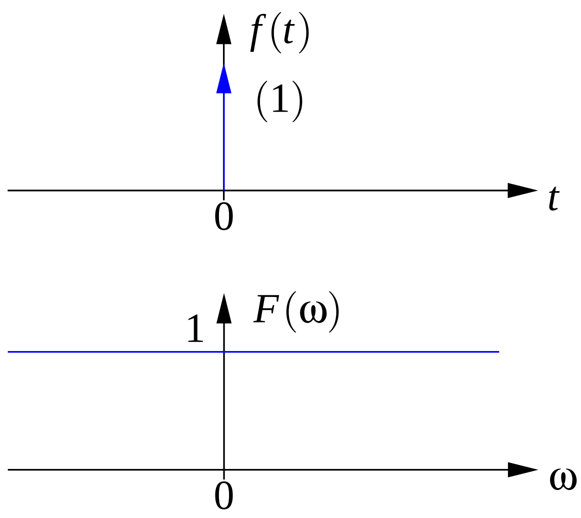
Example: Unit Impulse Function

$$f(t) = \delta(t)$$

Fourier Transform:

$$\mathcal{F}\{\delta(t)\} = \int_{-\infty}^{\infty} \delta(t) \exp(-j\omega t) dt$$

$$= \exp(-j\omega 0) = 1$$



Example: Complex Exponential Function

$$f(t) = \exp(j\omega_0 t)$$

The only frequency component present in the signal is ω_0.

Inverse Fourier Transform:

$$\mathcal{F}^{-1}\{\delta(\omega - \omega_0)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0)\exp(j\omega t)d\omega = \frac{1}{2\pi}\exp(\omega_0 t)$$

$$\exp(j\omega_0 t) \circ\!\!-\!\!\bullet\; 2\pi\delta(\omega - \omega_0)$$

3.2.2 Properties of the Fourier Transform

- **Linearity / Superposition:**

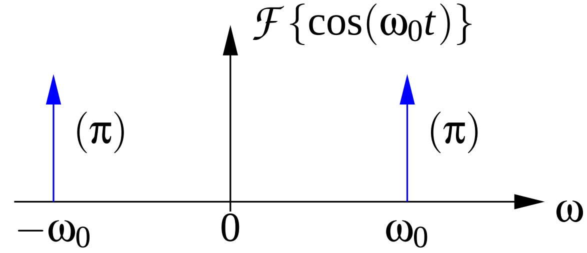
$$k_1 f_1(t) + k_2 f_2(t) \circ\!\!-\!\!\bullet\; k_1 F_1(\omega) + k_2 F_2(\omega)$$

where

$$f_1(t) \circ\!\!-\!\!\bullet\; F_1(\omega), \qquad f_2(t) \circ\!\!-\!\!\bullet\; F_2(\omega), \qquad k_1, k_2 : \text{ arbitray constants}$$

Example: Sinusoidal Signals

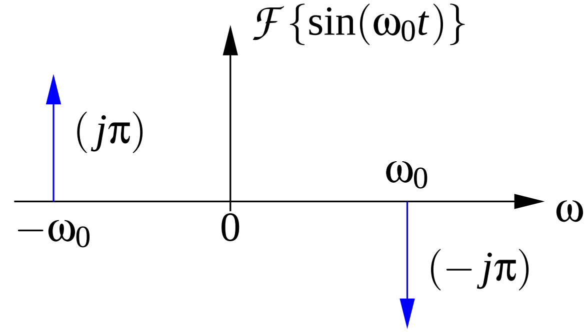
$$\cos(\omega_0 t) = \frac{1}{2}\exp(j\omega_0 t) + \frac{1}{2}\exp(-j\omega_0 t)$$



$$\exp(j\omega_0 t) \circ\!\!-\!\!\bullet\; 2\pi\delta(\omega - \omega_0)$$

$$\exp(-j\omega_0 t) \circ\!\!-\!\!\bullet\; 2\pi\delta(\omega + \omega_0)$$

$$\mathcal{F}\{\cos(\omega_0 t)\} = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



$$\sin(\omega_0 t) = \frac{1}{2j}\exp(j\omega_0 t) - \frac{1}{2j}\exp(-j\omega_0 t)$$

$$\mathcal{F}\{\sin(\omega_0 t)\} = -j\pi\delta(\omega - \omega_0) + j\pi\delta(\omega + \omega_0)$$

- **Complex Conjugate:**

$$f(t) \circ\!\!-\!\!\bullet\; F(\omega), \qquad\qquad f^*(t) \circ\!\!-\!\!\bullet\; F^*(-\omega)$$

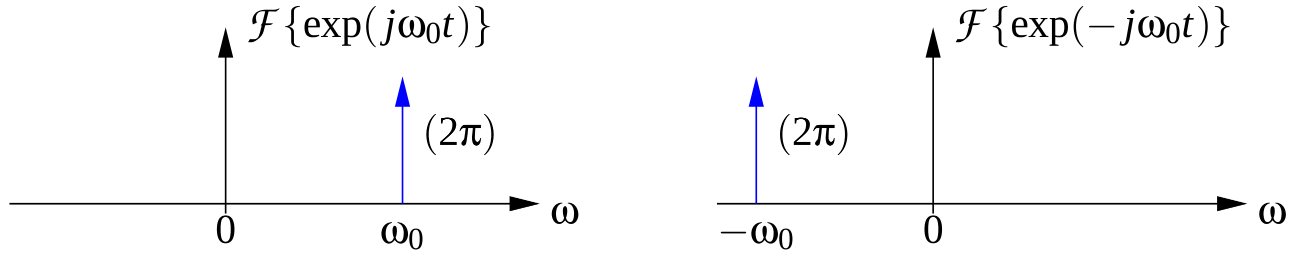
Proof:

$$\mathcal{F}\{f^*(t)\} = \int\limits_{-\infty}^{\infty} f^*(t)\exp(-j\omega t)dt = \left[\int_{-\infty}^{\infty} f(t)\exp(j\omega t)dt\right]^* = F^*(-j\omega)$$

Example: Complex Exponential Function

$$\exp(j\omega_0 t) \circ\!\!-\!\!\bullet\; 2\pi\delta(\omega - \omega_0)$$

$$[\exp(j\omega_0 t)]^* = \exp(-j\omega_0 t) \circ\!\!-\!\!\bullet\; 2\pi\delta(-\omega - \omega_0) = 2\pi\delta(\omega + \omega_0)$$



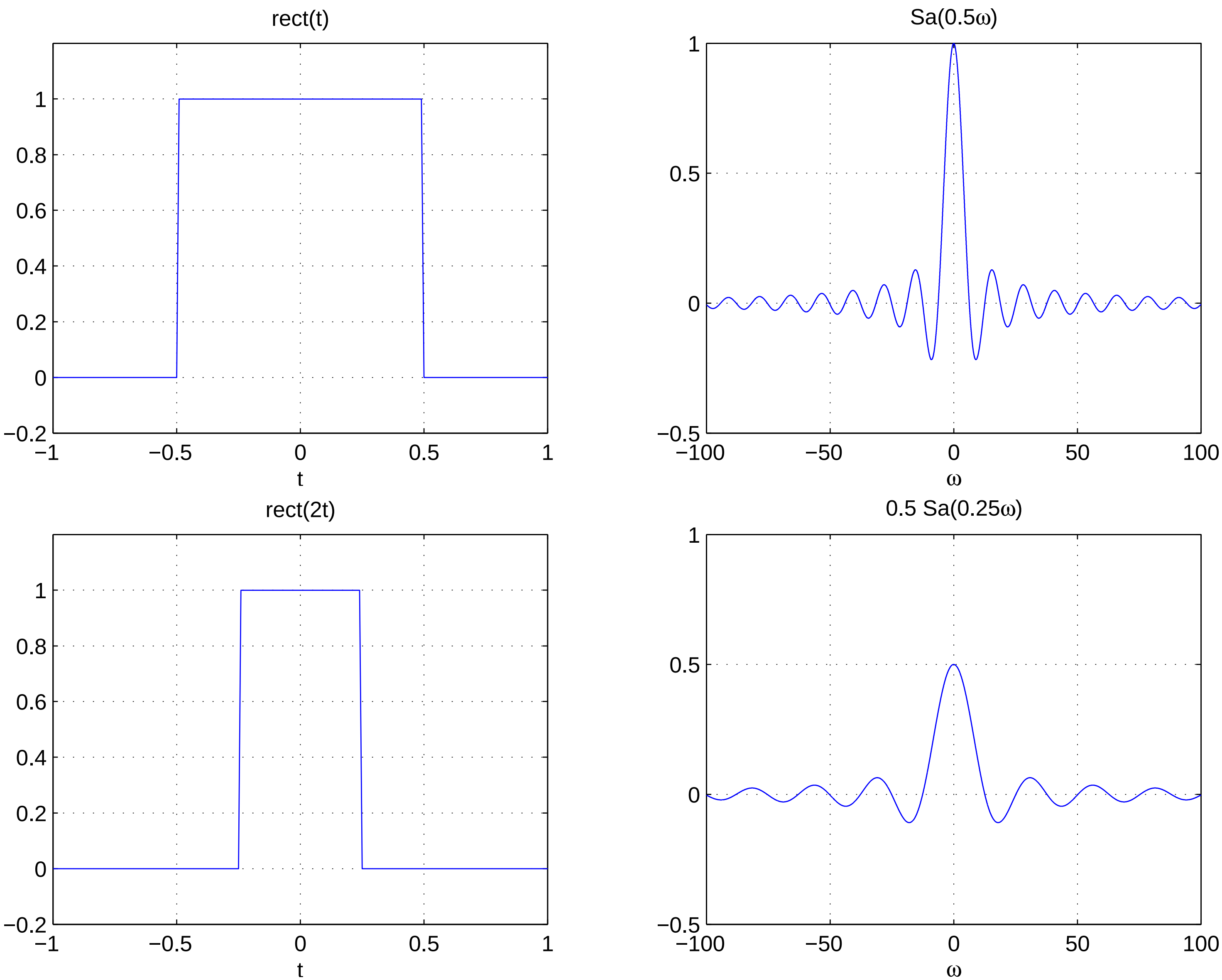
- **Coordinate Scaling (Reciprocal Spreading)**

$$f(at) \circ\!\!-\!\!\bullet\; \frac{1}{|a|} F\left(\frac{\omega}{a}\right), \qquad a: \text{ real valued constant}$$

For $a = -1$:

$$f(-t) \circ\!\!-\!\!\bullet\; F(-\omega)$$

Example: Gate Function



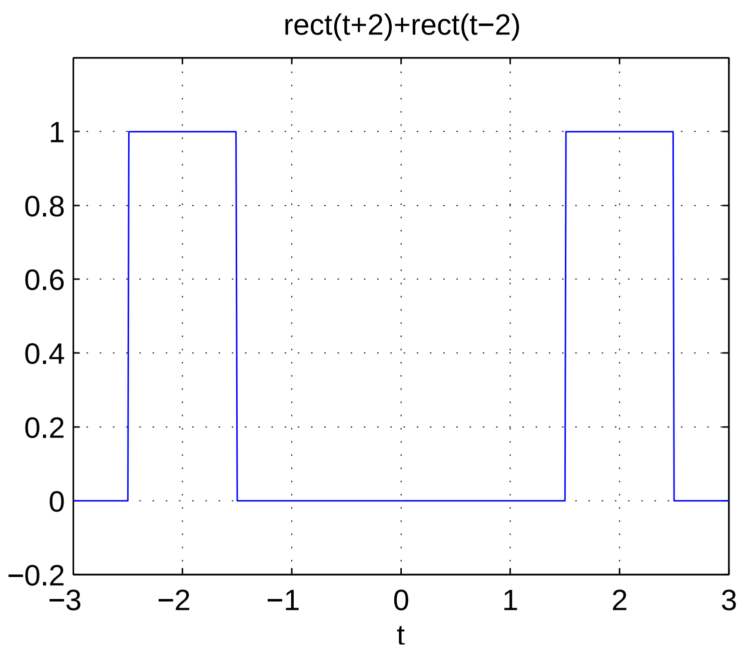
- **Time Shifting (Delay):**

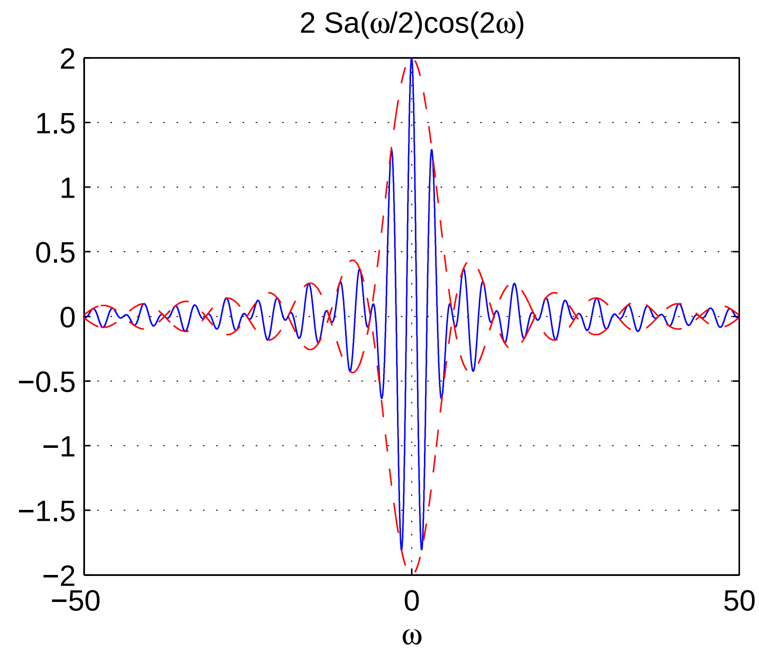
$$f(t-t_0) \circ\!\!-\!\!\bullet\; F(\omega)\exp(-j\omega t_0), \qquad t_0: \text{ real valued constant}$$

Example: Shifted Rectangular Pulses

$$f(t) = \text{rect}(t+2) + \text{rect}(t-2), \qquad \text{rect}(t) \circ\!\!-\!\!\bullet\; \text{Sa}(\omega/2)$$

$$f(t) \circ\!\!-\!\!\bullet\; \text{Sa}(\omega/2)\big(\exp(j2\omega) - \exp(-j2\omega)\big) = \text{Sa}(\omega/2)\cdot 2\cos(2\omega)$$





- **Frequency Shifting (Modulation)**

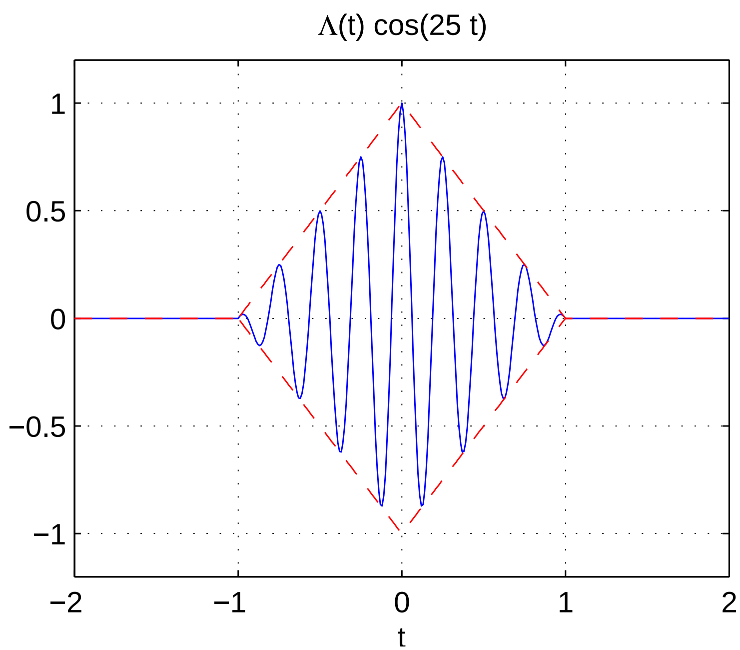
$$f(t) \cdot e^{j\omega_0 t} \circ\!\!-\!\!\bullet\ F(\omega - \omega_0)$$

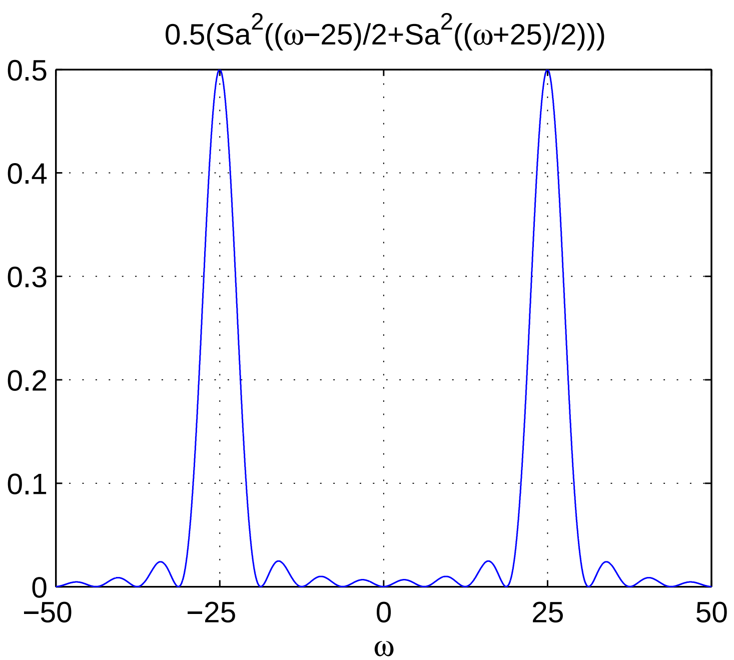
Example: Amplitude Modulation of a Triangular Pulse

$$f(t) = \Lambda(t) \cdot \cos(\omega_0 t) = \frac{1}{2}\Lambda(t)(\exp(j\omega_0 t) + \exp(-j\omega_0 t))$$

$$\Lambda(t) \circ\!\!-\!\!\bullet\ \mathrm{Sa}^2(\omega/2)$$

$$f(t) \circ\!\!-\!\!\bullet\ \frac{1}{2}\left(\mathrm{Sa}^2((\omega - \omega_0)/2) + \mathrm{Sa}^2((\omega + \omega_0)/2)\right)$$





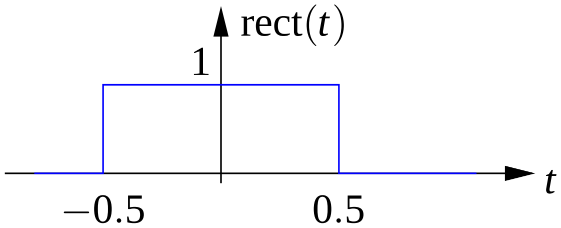
- **Differentiation**

$$\frac{d}{dt} f(t) \circ\!\!-\!\!\bullet\; j\omega \cdot F(\omega)$$

Example: Gate Function

What is the Fourier Transform of

$$\frac{d}{dt}\text{rect}(t) = \delta(t+0.5) - \delta(t-0.5)$$

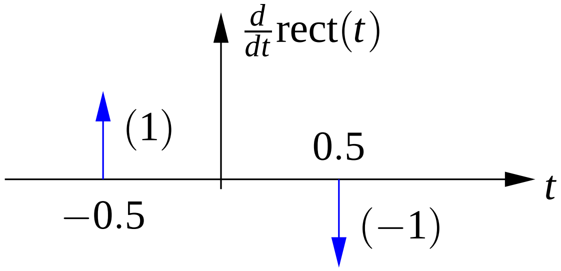


With:

$$\text{rect}(t) \circ\!\!-\!\!\bullet\; \text{Sa}(\omega/2)$$

$$\frac{d}{dt}\text{rect}(t) \circ\!\!-\!\!\bullet\; j\omega\text{Sa}(\omega/2) = j\omega\frac{\sin(\omega/2)}{\omega/2}$$

$$= 2j\sin(\omega/2)$$



Compare to:

$$\delta(t+0.5) - \delta(t-0.5) \circ\!\!-\!\!\bullet\; \exp(j\omega/2) - \exp(-j\omega/2) = 2j\sin(\omega/2)$$

- **Integration**

$$\int_{-\infty}^{t} f(\tau)\, d\tau \;\circ\!\!-\!\!\bullet\; \frac{1}{j\omega}\, F(\omega) + \pi\, F(0)\, \delta(\omega)$$

Example: Unit Step Function $u(t)$:

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau, \qquad \delta(t) \;\circ\!\!-\!\!\bullet\; 1$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)\, d\tau \;\circ\!\!-\!\!\bullet\; \frac{1}{j\omega} + \pi\, \delta(\omega)$$

3.2.3 Symmetry of the Fourier Transform

$$f(t) = f_r(t) + j f_i(t) = f_{r,e}(t) + f_{r,o}(t) + j(f_{i,e}(t) + f_{i,o}(t))$$

$f_r(t)$: real part of $f(t)$ $\quad$ $f_i(t)$: imaginary part of $f(t)$

$f_{r,e}(t)$ / $f_{i,e}(t)$: even symmetry part of the real / imaginary part of $f(t)$

$f_{r,o}(t)$ / $f_{i,o}(t)$: odd symmetry part of the real / imaginary part of $f(t)$

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\exp(-j\omega t)dt = \int_{-\infty}^{\infty} f(t)\left(\cos(\omega t) - j\sin(\omega t)\right)dt$$

$$= \int_{-\infty}^{\infty} \left(f_{r,e}(t) + f_{r,o}(t) + j f_{i,e}(t) + j f_{i,o}(t)\right)\left(\cos(\omega t) - j\sin(\omega t)\right)dt$$

$$= \underbrace{2\int_0^{\infty} f_{r,e}(t)\cos(\omega t)dt}_{\text{even symmetry, real}} - \underbrace{2j\int_0^{\infty} f_{r,o}(t)\sin(\omega t)dt}_{\text{odd symmetry, imaginary}}$$

$$+ \underbrace{2j\int_0^{\infty} f_{i,e}(t)\cos(\omega t)dt}_{\text{even symmetry, imaginary}} + \underbrace{2\int_0^{\infty} f_{i,o}(t)\sin(\omega t)dt}_{\text{odd symmetry, real}}$$

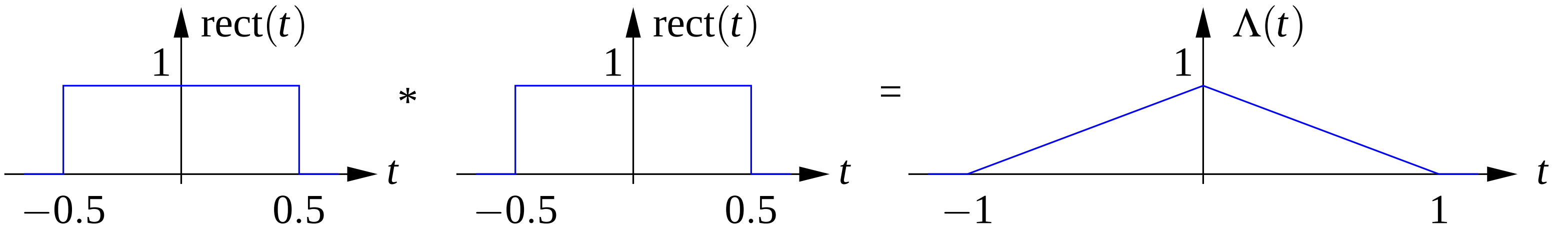
$f(t)$	$F(\omega)$
real, even	real, even
real, odd	imaginary, odd
imaginary, even	imaginary, even
imaginary, odd	real, odd
real	even real part, odd imaginary part
imaginary	odd real part, even imaginary part
even real part, odd imaginary part	real
odd real part, even imaginary part	imaginary
even real part, even imaginary part	even real part, even imaginary part
odd real part, odd imaginary part	odd real part, odd imaginary part

3.2.4 Convolution

- **Time Convolution:**

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau) d\tau \circ\!\!-\!\!\bullet\; F_1(\omega) \cdot F_2(\omega)$$

Example: Triangular Function



$$\Lambda(t) = \text{rect}(t) * \text{rect}(t) \circ\!\!-\!\!\bullet\; \text{Sa}(\omega/2) \cdot \text{Sa}(\omega/2) = \text{Sa}^2(\omega/2)$$

Example: Unit Impulse Function

$$f(t) * \delta(t) \circ\!\!-\!\!\bullet\; F(\omega) \cdot 1 \bullet\!\!-\!\!\circ\; f(t)$$

$$f(t) * \delta(t-t_0) \circ\!\!-\!\!\bullet\; F(j\omega) \cdot 1 \cdot e^{-j\omega t_0} \bullet\!\!-\!\!\circ\; f(t-t_0)$$

- **Frequency Convolution:**

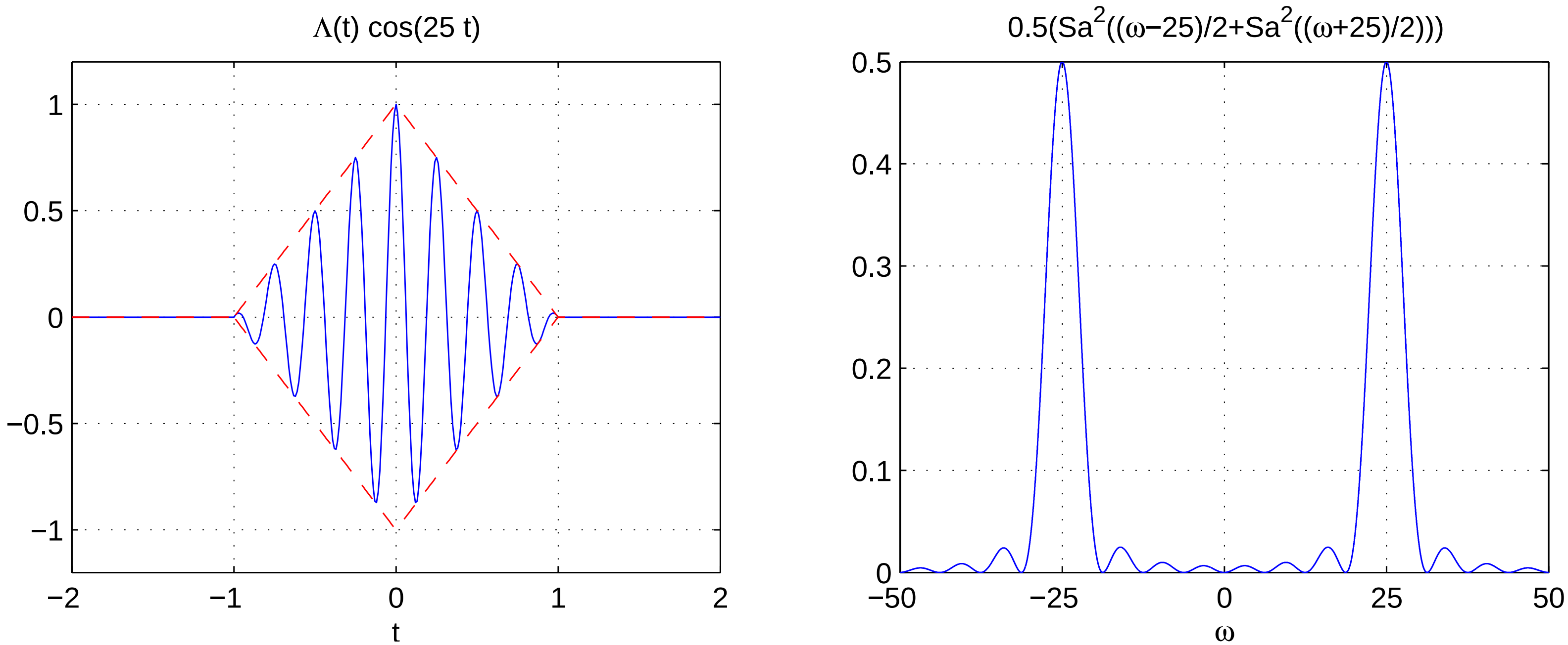
$$F_1(\omega) * F_2(\omega) = \int_{-\infty}^{\infty} F_1(\nu) \cdot F_2(\omega - \nu)\, d\nu \bullet\!\!-\!\!\circ\; 2\pi \cdot f_1(t) \cdot f_2(t)$$

Example: Amplitude Modulation of a Triangular Pulse

$$f(t) = \Lambda(t) \cdot \cos(\omega_0 t) = \frac{1}{2}\Lambda(t)(\exp(j\omega_0 t) + \exp(-j\omega_0 t))$$

$$\Lambda(t) \circ\!\!-\!\!\bullet\; \mathrm{Sa}^2(\omega/2), \qquad \exp(\pm j\omega_0 t) \circ\!\!-\!\!\bullet\; 2\pi\delta(\omega \mp \omega_0)$$

$$f(t) \circ\!\!-\!\!\bullet\; \frac{1}{2\pi}\mathrm{Sa}^2(\omega/2) * (\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0))$$



3.2.5 Parseval's Theorem for Energy Signals

$$E = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega$$

Example:

$$f(t) = \mathsf{Sa}(t/2) = \frac{\sin(t/2)}{t/2} \circ\!\!-\!\!\bullet\; 2\pi \mathsf{rect}(\omega) = F(\omega)$$

$$E = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \int_{-\infty}^{\infty} \frac{\sin^2(t/2)}{(t/2)^2} \, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 \, d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} 4\pi^2 \mathsf{rect}^2(\omega) \, d\omega$$

$$= 2\pi \int_{-0.5}^{0.5} 1 \; d\omega = 2\pi$$

3.2.6 Fourier Transform of Periodic Signals

Exponential Fourier Series representation of a periodic signal:

$$f_T(t) = \sum_{k=-\infty}^{\infty} F_k \exp(jk\omega_0 \, t)$$

Fourier Transform of a periodic signal $f_T(t)$:

$$\mathcal{F}\{f_T(t)\} = \mathcal{F}\{\sum_{k=-\infty}^{\infty} F_k \exp(jk\omega_0 t)\} = \sum_{k=-\infty}^{\infty} F_k \, \mathcal{F}\{\exp(jk\omega_0 t)\}$$

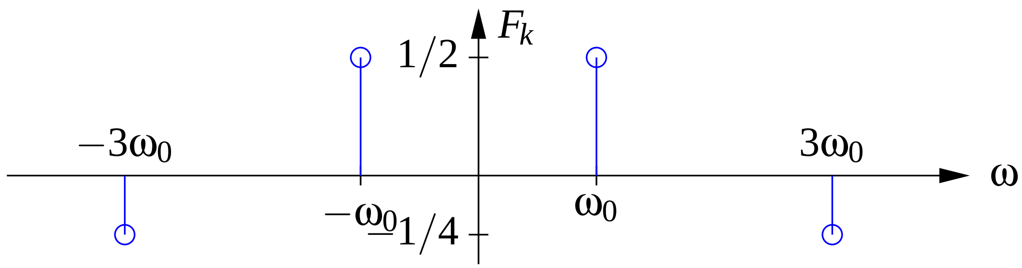
$$= \sum_{k=-\infty}^{\infty} F_k 2\pi\delta(\omega - k\omega_0)$$

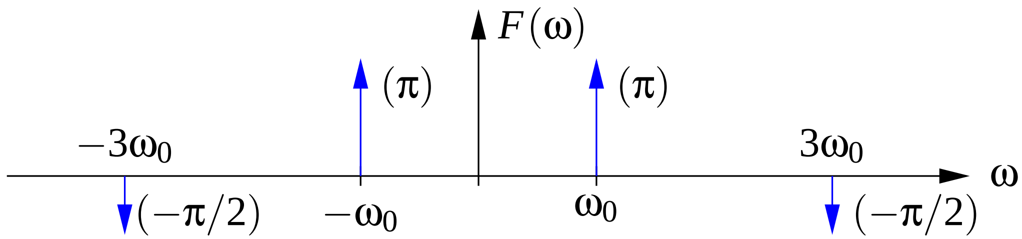
Example: Sum of Two Cosine Signals

$$f_T(t) = \cos(\omega_0 t) - 0.5\cos(3\omega_0 t)$$

$$= 0.5\big(\exp(j\omega_0 t) + \exp(-j\omega_0 t)\big) - 0.25\big(\exp(j3\omega_0 t) + \exp(-j3\omega_0 t)\big)$$

$$F_{-1} = F_1 = 0.5$$
$$F_{-3} = F_3 = -0.25$$

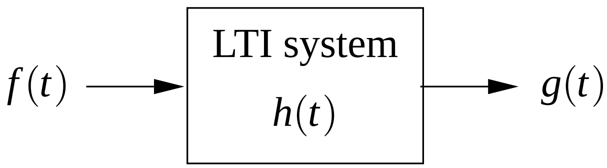

F_k
$1/2$
$-3\omega_0$
$3\omega_0$
ω
$-\omega_0$
$-1/4$
ω_0


$F(\omega)$
(π)
(π)
$-3\omega_0$
$3\omega_0$
ω
$(-\pi/2)$
$-\omega_0$
ω_0
$(-\pi/2)$

3.3 Frequency Domain Description of LTI Systems

3.3.1 Frequency Response

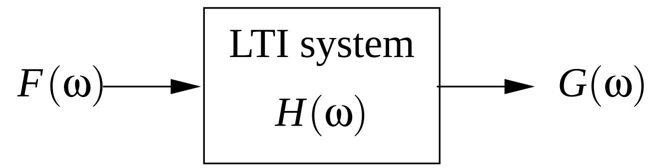
Time Domain:



$h(t)$: Impulse Response

$$g(t) = f(t) * h(t)$$

Frequency Domain:

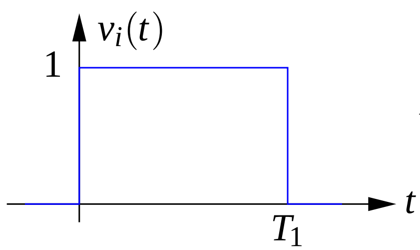


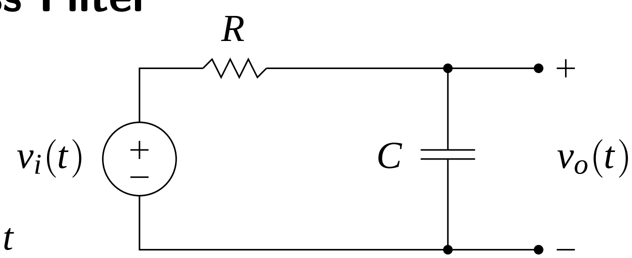
$H(\omega)$: Frequency Response

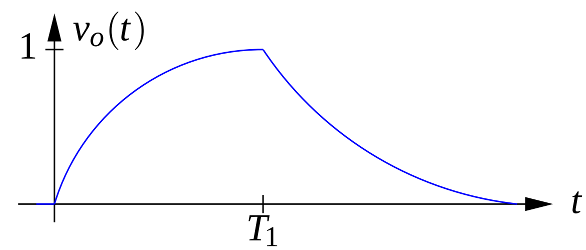
$$G(\omega) = F(\omega) \cdot H(\omega)$$

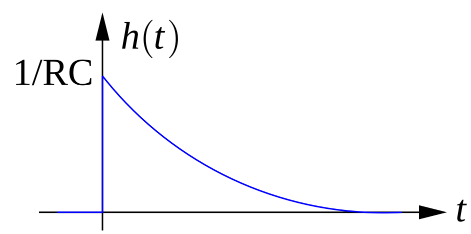
An LTI system does not generate new frequency components.

Example: RC Lowpass Filter

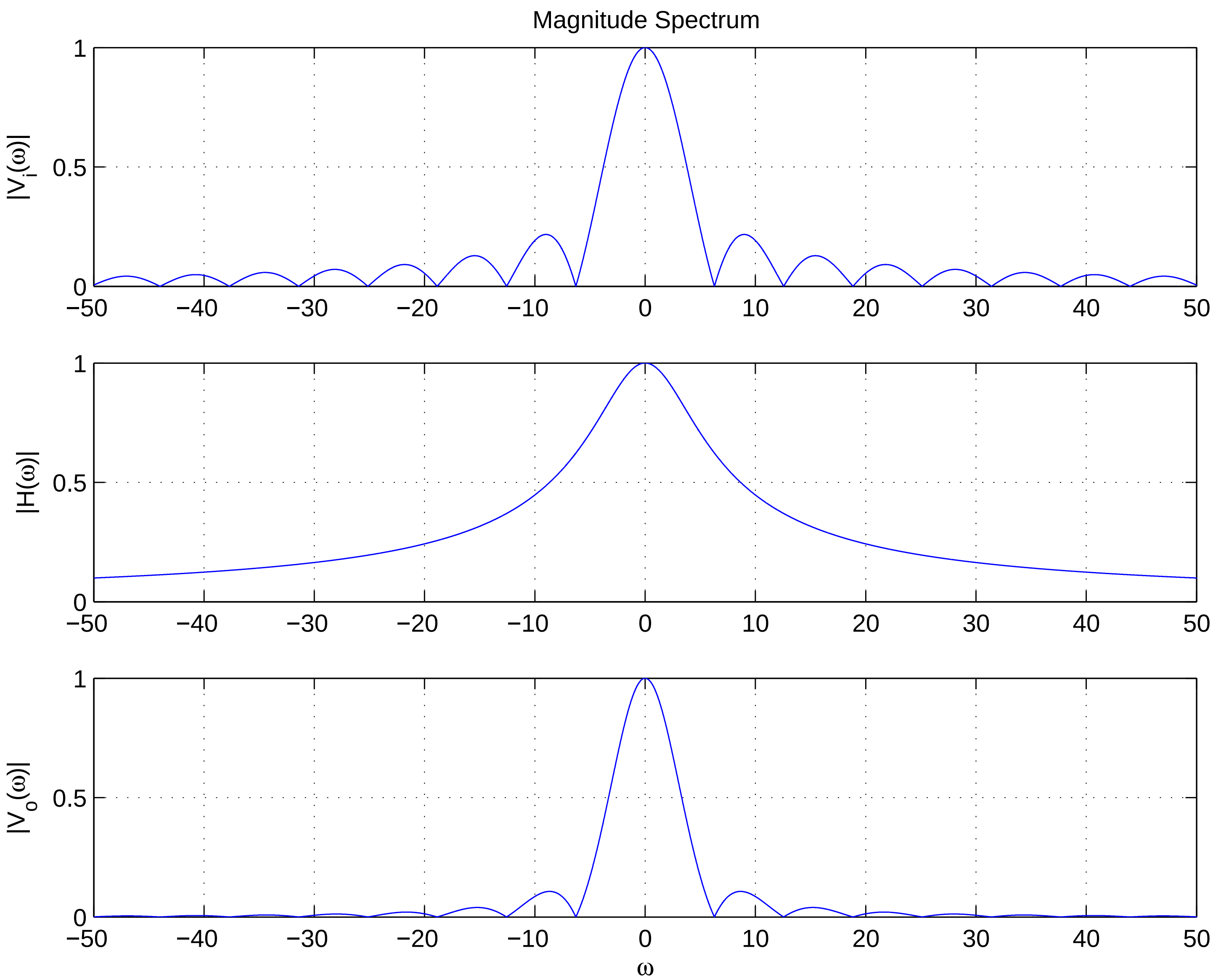


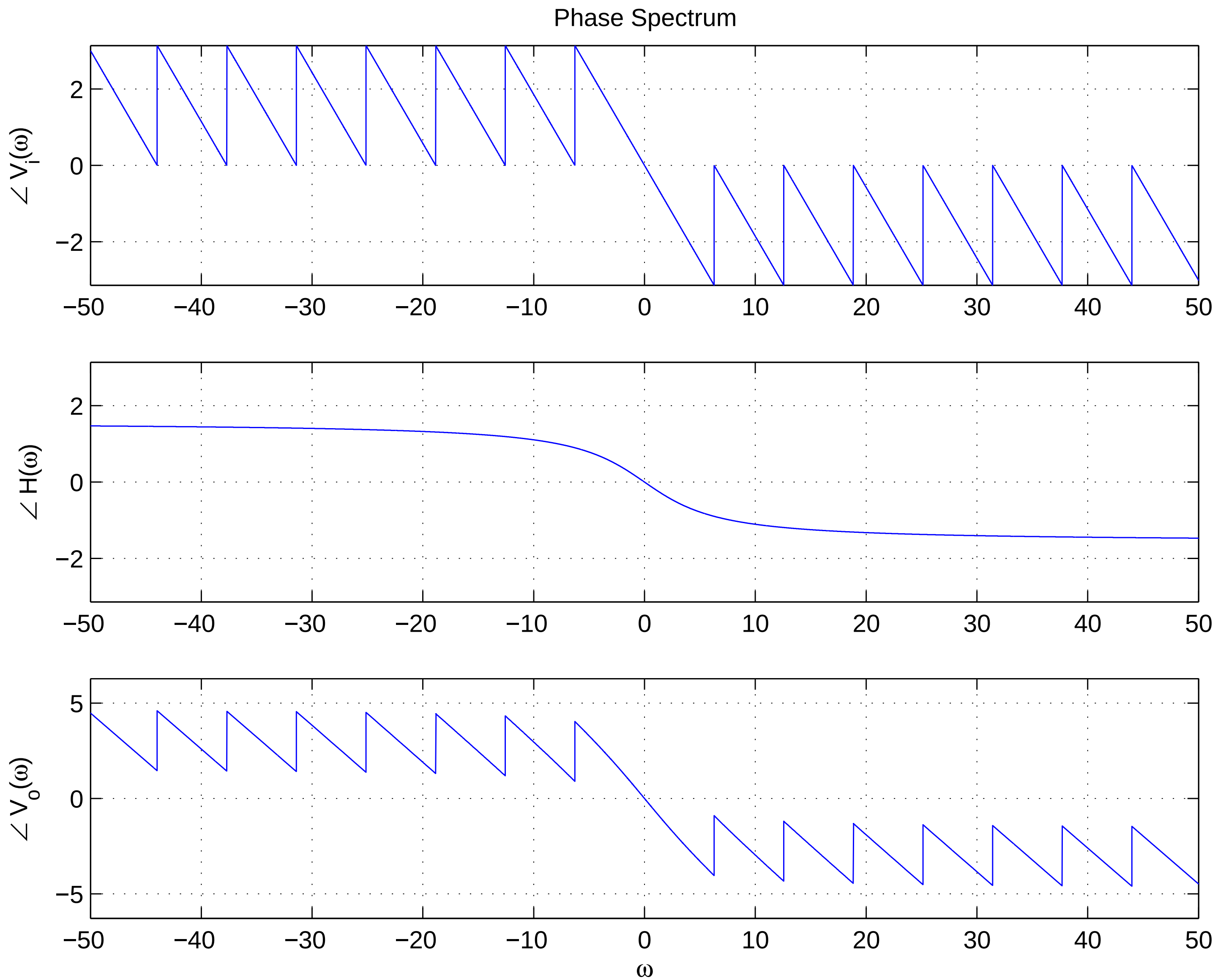






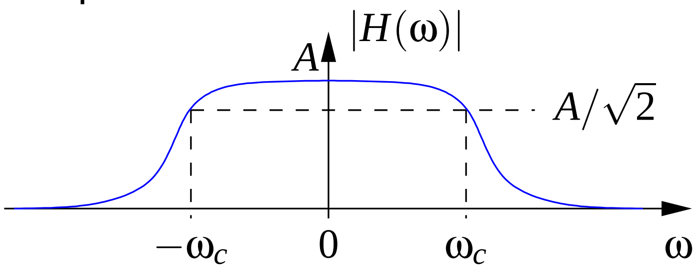
For $T_1 = 1$, RC=10:


Magnitude Spectrum
|V_i(ω)|
|H(ω)|
|V_o(ω)|
ω
0
0.5
1
−50
−40
−30
−20
−10
0
10
20
30
40
50


Phase Spectrum
∠ V_i(ω)
∠ H(ω)
∠ V_o(ω)
ω
2
0
−2
5
−5
−50
−40
−30
−20
−10
10
20
30
40
50

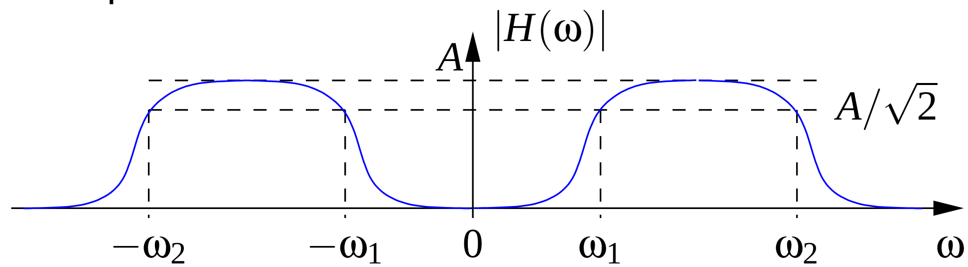
3.3.2 Bandwidth of Frequency Selective Systems

Lowpass Filter:



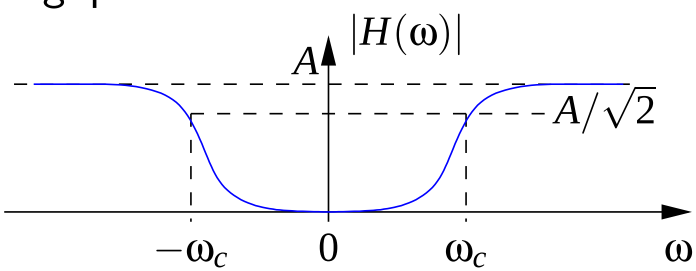
Bandwidth: $B = \omega_c$

Bandpass Filter:

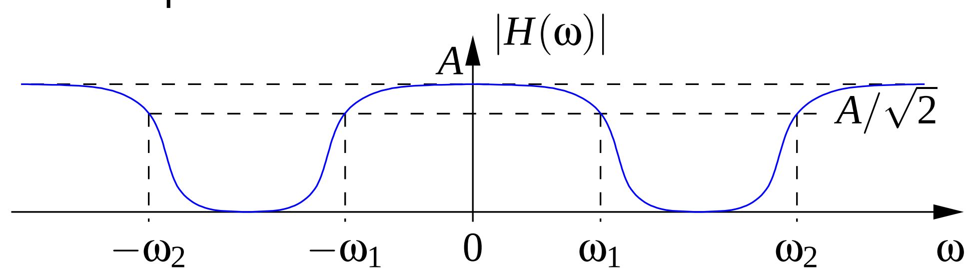


Bandwidth: $B = \omega_2 - \omega_1$

Highpass Filter:



Bandstop Filter:

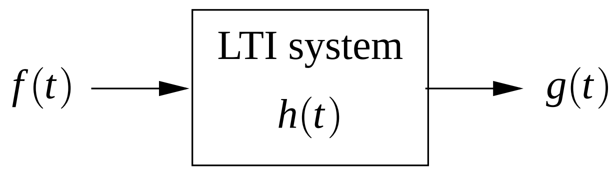


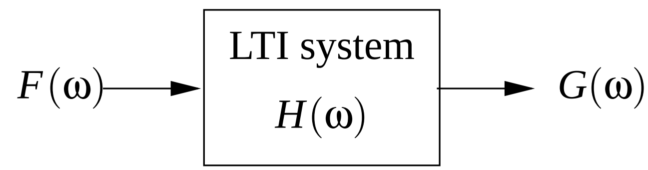
Only positive frequencies are counted for the bandwidth.

3.3.3 Distortionless Transmission

A transmission system is called distortionless, if the output signal $g(t)$ is a scaled and deltayed copy of the input signal $f(t)$:

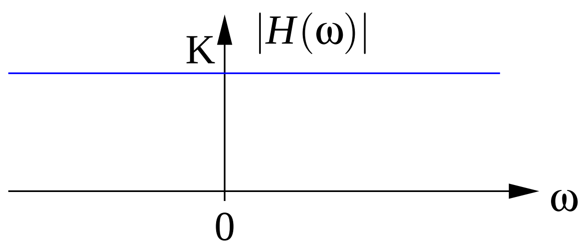
$$g(t) = K \cdot f(t - t_0) \qquad G(\omega) = K \cdot F(\omega)\exp(-j\omega t_0)$$

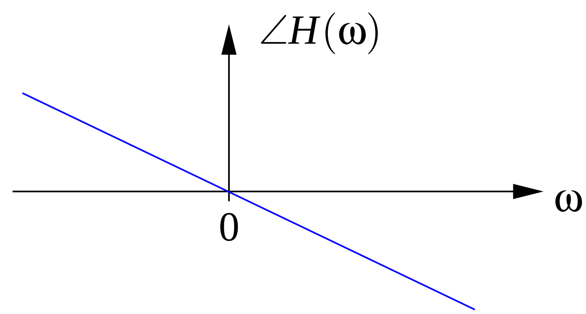




System Frequency Response:

$$H(\omega) = K \cdot \exp(-j\omega t_0)$$





The transmission system must have a constant magnitude response and its phase shift must be linear with frequency (resulting in the same delay of all frequency components of the input).